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SEMICONINUOUS, QUASI-COMPACT AND RELATED MULTIFUNCTIONS

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CHAPTER I

INTRODUCTION

The central theme of this work is the study of some topological concepts for set-valued functions (hereafter referred to as functions) which have been well studied for single-valued functions.

In Chapter 2 the notions of upper and lower semicontinuity are discussed for functions and these notions are related to upper and lower semicontinuous decompositions and real-valued single-valued functions. An extension theorem which generalizes the Tietze Extension Theorem is proved for functions with continuum-valued point images. A condition which is necessary and sufficient for a space to be a k -space is given in terms of functions defined on the space. This result was known for single-valued functions.

Chapter 3 is mainly concerned with quasi-compactness of functions and decompositions induced by such functions. In the past, both of these concepts have apparently been defined only for semi-single-valued functions. An order is defined for a function, where every semi-single-valued function has order one, and many known results for semi-single-valued functions are extended to functions, sometimes under the restriction that the function have finite order. Some theorems of E. Duda concerning the relationship between reflexive closed mappings (single-valued continuous functions), reflexive compact mappings, and upper semicontinuous decompositions are extended to functions, with no need for any continuity conditions.

In Chapter 4 upper and lower semicontinuity, compactness, having a closed graph, and a seemingly new concept, semicontinuity, are studied in relationship to one another. Many results of R. V. Fuller, for single-valued functions, and of G. T. Whyburn, for functions, result as corollaries to the main theorems. Equivalent conditions are found for having a closed graph, being semi-closed, compactness, upper semicontinuity, and semicontinuity in terms of the location of cluster points of certain filterbases.

CHAPTER II

PRELIMINARY RESULTS

This chapter contains basic definitions and comments which are used throughout the remainder of this work. Upper semicontinuous and lower semicontinuous functions are discussed, and these concepts are related to similar ones for decompositions and real-valued single-valued functions.

1. Definitions and General Comments

If \bar{X} and \bar{Y} are topological spaces, a function T from \bar{X} into \bar{Y} , denoted $T:\bar{X} \rightarrow \bar{Y}$, will always mean a relationship which assigns to each element in \bar{X} a non-empty subset of \bar{Y} . These functions are referred to as relations or multifunctions in much of the literature. If a function is such that the non-empty intersection of two point images implies their equality, it will be called semi-single-valued (ssv); the term "non-mingled" is also in common usage. If point images are always one point subsets the function will be called single-valued.

(1.1) Definition: If $T:\bar{X} \rightarrow \bar{Y}$ is a function, the function $T^{-1}:T(\bar{X}) \rightarrow \bar{X}$ defined by $T^{-1}(y) = \{x \text{ in } \bar{X}: y \text{ is in } T(x)\}$ is called the inverse of T . If A is any non-empty subset of \bar{Y} , then $T^{-1}(A) = \{x \text{ in } \bar{X}: T(x) \cap A \text{ is non-empty}\}$.

For any set \bar{X} , the power set of \bar{X} (i.e., the set of all subsets of \bar{X}) will be denoted by $P(\bar{X})$.

(1.2) Definition: If $T: \bar{X} \rightarrow \bar{Y}$ is a function then the function $T^+: P(\bar{Y}) \rightarrow P(\bar{X})$ defined by $T^+(B) = \{x \text{ in } \bar{X}: T(x) \text{ is contained in } B\}$ is called the lower inverse of T .

Note that $T^+(B)$ can be empty and in general, $T^+(B)$ is contained in $T^{-1}(B)$.

(1.3) Theorem: If $T: \bar{X} \rightarrow \bar{Y}$ is any function, V is a subset of \bar{Y} and U_i are subsets of \bar{X} , then

- (a) $T(\Sigma U_i) = \Sigma T(U_i)$,
- (b) $T(\Pi U_i) \subset \Pi T(U_i)$,
- (c) for V in $T(\bar{X})$, $TT^+(V) \subset V \subset TT^{-1}(V)$, and
- (d) $T^{-1}(\bar{Y} - V) = \bar{X} - T^+(V)$.

Proof: (a): Let y be in $T(\Sigma U_i)$; then there is a U_i and an x_i in U_i so that y is in $T(x_i) \subset T(U_i) \subset \Sigma T(U_i)$. If y is in $\Sigma T(U_i)$, there is a U_i and an x_i so that y is in $T(x_i)$ and x_i is in U_i . But then y is in $T(\Sigma U_i)$, which proves (a).

(b): Let y be in $T(\Pi U_i)$; then there is an x in each U_i so that y is in $T(x) \subset T(U_i)$ for all i . Hence y is in $\Pi T(U_i)$ and (b) is proved.

(c): Let y be in $TT^+(V)$; then there is an x in $T^+(V)$ so that y is in $T(x)$ and $T(x) \subset V$, by the definition of the lower inverse. Thus y is in V proving the first assertion.

Let z be in V ; then there is an x in $T^{-1}(V)$ so that x is in $T^{-1}(z)$. Since x is in $T^{-1}(z)$ if and only if z is in $T(x)$, z is in $TT^{-1}(V)$.

(d): $T^{-1}(\bar{Y} - V) = \{x \text{ in } \bar{X}: T(x) \cdot (\bar{Y} - V) \text{ is non-empty}\} = \bar{X} - \{x \text{ in } \bar{X}: T(x) \subset V\} = \bar{X} - T^+(V)$.

Whyburn [9, Section 1, p. 343] has proved the following theorem for ssv functions.

(1.4) Theorem: A function T is ssv if and only if one of the following conditions is satisfied:

- (a) $TT^{-1}T = T$ on \bar{X} .
- (b) $T^{-1}TT^{-1} = T^{-1}$ on $T(\bar{X})$.
- (c) For each image set B in $T(\bar{X})$, $TT^{-1}(B) = B$.
- (d) For each inverse set A in \bar{X} , $T^{-1}T(A) = A$.
- (e) For any inverse set A in \bar{X} , $T(\bar{X} - A) = T(\bar{X}) - T(A)$.
- (f) T^{-1} is ssv.

A set B in $T(\bar{X})$ is an image set for T if there is a set A in \bar{X} so that $B = T(A)$; an inverse set is an image set for T^{-1} .

(1.5) Definition: The function $T: \bar{X} \rightarrow \bar{Y}$ is upper semicontinuous (usc) at x in \bar{X} if and only if for every neighborhood V of $T(x)$ there is a neighborhood U of x so that $T(U) \subset V$. The function T is lower semicontinuous (lsc) at $x \in \bar{X}$ if and only if for every open set V in \bar{Y} so that $V \cap T(x)$ is non-empty there is a neighborhood U of x so that for every x' in U , $T(x') \cap V$ is non-empty. If T is usc and lsc at x , T is continuous at x . If T is usc (lsc) for every x in \bar{X} then T is usc on \bar{X} (lsc on \bar{X}).

If F^* is a family of sets, the collection $\{T(F): F \text{ in } F^*\}$ is denoted $T(F^*)$. A filter F^* on a topological space \bar{X} is a non-empty collection of non-empty subsets of \bar{X} satisfying the following conditions:

- (1) for F_1, F_2 in F^* there is an F_3 in F^* so that $F_3 \subset F_1 \cap F_2$, and
- (2) any set S in \bar{X} containing an element of F^* is in F^* .

If F^* satisfies condition (1) it is a filterbase on \bar{X} . If S is any non-empty subset of \bar{X} , $N^*(S)$ denotes the neighborhood filter of S ; i.e., the filter of those subsets of \bar{X} containing an open set containing S .

The symbol $[B^*]$ stands for the filter generated by the filterbase B^* . If A^* and B^* are filterbases, A^* is finer than B^* if every element of B^* contains an element of A^* . The following are some definitions and facts about filters and filterbases on topological spaces (Berge [1]):

(a) A point x in \bar{X} is a limit point of the filter (filterbase) F^* if F^* is finer than $N^*(x)$; F^* is also said to converge to x , written $F^* \rightarrow x$.

(b) A point x in \bar{X} is a cluster point of the filter (filterbase) F^* if it lies in the closure in \bar{X} of all elements in F^* .

(c) A point is a cluster point of a filterbase F^* if and only if every neighborhood of the point meets every element of F^* .

(d) A point x is a cluster point of a filter (filterbase) F^* if and only if there is a filter (filterbase) finer than F^* which converges to x .

(e) Let B^* be a filterbase on a non-empty subset A of \bar{X} ; then every cluster point of B^* belongs to the closure of A . Conversely, every point of the closure of A is a limit point of a filterbase on A .

(f) If F^* is a filterbase of compact sets, then F^* has a cluster point.

The following lemmas will be used extensively in the sequel. The first two were proved by Whyburn [8, comments (a), (b), p. 1496]. In every case $T: \bar{X} \rightarrow \bar{Y}$ is a function.

(1.6) Lemma: For any filterbase M^* on \bar{X} , $N^* = T(M^*)$ is a filterbase on \bar{Y} .

(1.7) Lemma: If M^* is a filterbase on \bar{X} and N^{**} is any filterbase finer than $N^* = T(M^*)$, then M^* and $T^{-1}(N^{**}) = M^{**}$ have a common finer filterbase $P^* = \{M \cdot M' : M \text{ is in } M^*, M' \text{ is in } M^{**}\}$.

(1.8) Lemma: Let B^* be a filterbase and C a set so that $B \cdot C$ is non-empty for every B in B^* . Then $A^* = \{B \cdot C : B \text{ is in } B^*\}$ is a filterbase finer than B^* .

Proof: Let $A_1 = B_1 \cdot C$, $A_2 = B_2 \cdot C$ be in A^* . Since B^* is a filterbase there is an element B_3 in B^* so that $B_3 \subset B_1 \cdot B_2$. Let $A_3 = B_3 \cdot C$; then $A_3 = B_3 \cdot C \subset (B_1 \cdot B_2) \cdot C \subset (B_1 \cdot C) \cdot (B_2 \cdot C) = A_1 \cdot A_2$ and thus $A_1 \cdot A_2$ is non-empty and also contains another element of A^* , proving that A^* is a filterbase. To see that A^* is finer than B^* , let B be in B^* ; then $B \cdot C \subset B$ and $B \cdot C$ is in A^* .

2. Upper and Lower Semicontinuity

The following results bring together many of the equivalent definitions of usc and lsc functions which exist in the literature.

(2.1) Theorem: The following statements are equivalent for the function $T: \bar{X} \rightarrow \bar{Y}$:

- (a) T is usc at x .
- (b) $N^*(T(x)) \subset [T(N^*(x))]$.
- (c) If V is in $N^*(T(x))$, $T^+(V)$ is in $N^*(x)$.

Proof: (a) \rightarrow (b): Let N be in $N^*(T(x))$; then there is an M in $N^*(x)$

so that $T(M) \subset N$, which places N in $[T(N^*(x))]$.

(b) \rightarrow (c): Let V be in $N^*(T(x))$; then V is in $[T(N^*(x))]$ and hence there is a U in $N^*(x)$ so that $T(U) \subset V$. Therefore $U \subset T^+T(U) \subset T^+(V)$, which places $T^+(V)$ in $N^*(x)$.

(c) \rightarrow (a): Let V be in $N^*(T(x))$ and $U = T^+(V)$ be in $N^*(x)$; then $T(U) = TT^+(V) \subset V$ by Theorem (1.3) (c) and T is usc at x .

(2.2) Theorem: The following statements are equivalent for the function $T: \bar{X} \rightarrow \bar{Y}$:

- (a) T is lsc at x .
- (b) For every y in $T(x)$, $T^{-1}(N^*(y)) \subset N^*(x)$.
- (c) For every open set V so that $V \cdot T(x)$ is non-empty, $T^{-1}(V)$ is in $N^*(x)$.

Proof: (a) \rightarrow (b): Let y be in $T(x)$, N in $N^*(y)$; then there is an M in $N^*(x)$ s. that for each p in M , $T(p) \cdot N$ is non-empty, or equivalently, p is in $T^{-1}(N)$. Hence $M \subset T^{-1}(N)$ and $T^{-1}(N)$ is thus a member of $N^*(x)$.

(b) \rightarrow (c): Let V be an open set so that $V \cdot T(x)$ is non-empty and let y be in $V \cdot T(x)$; then V is in $N^*(y)$ and by (b), $T^{-1}(V)$ is in $N^*(x)$.

(c) \rightarrow (a): Let V be an open set so that $V \cdot T(x)$ is non-empty; then $T^{-1}(V)$ is in $N^*(x)$ and the image of every point in $T^{-1}(V)$ intersects V nonvacuously. Thus T is lsc at x .

(2.3) Theorem: The following statements are equivalent for the function $T: \bar{X} \rightarrow \bar{Y}$:

- (a) T is usc on \bar{X}
- (b) The inverse of every set which is closed in $T(\bar{X})$ is closed.

(c) The lower inverse of every set which is open in $T(\bar{X})$ is open.

Proof: (a) \rightarrow (b): Let C be closed in $T(\bar{X})$ and x be in $\bar{X} - T^{-1}(C)$; then $T(x) \cdot C$ is empty and $T(\bar{X}) - C$ is in $N^*(T(x))$. From Theorem (2.1) (b), $T(\bar{X}) - C$ is in $[T(N^*(x))]$ and hence there is an N in $N^*(x)$ so that $T(N) \subset T(\bar{X}) - C$. Therefore $T(N) \cdot C$ is empty and $N \subset \bar{X} - T^{-1}(C)$, which shows that $\bar{X} - T^{-1}(C)$ is open and thus $T^{-1}(C)$ is closed.

(b) \rightarrow (c): Let U be open in $T(\bar{X})$; then $T(\bar{X}) - U$ is closed in $T(\bar{X})$. From Theorem (1.3) (d), $T^{-1}(T(\bar{X}) - U) = \bar{X} - T^+(U)$. From part (b), $\bar{X} - T^+(U)$ is closed and so $T^+(U)$ is open.

(c) \rightarrow (a): Let x be in \bar{X} and V in $N^*(T(x))$; then there is an open set U so that $T(x) \subset U \subset V$. From (c) $T^+(U)$ is open, and non-empty since x is in $T^+(U)$. Thus $T^+(U)$ is in $N^*(x)$ and $TT^+(U) \subset U$, and T is usc at x .

(2.4) Theorem: The following statements are equivalent for the function $T: \bar{X} \rightarrow \bar{Y}$:

(a) T is lsc on \bar{X} .

(b) The inverse of every set which is open in $T(\bar{X})$ is open.

(c) The lower inverse of every set which is closed in $T(\bar{X})$ is closed.

Proof: (a) \rightarrow (b): Let $U \subset T(\bar{X})$ be open in $T(\bar{X})$ and x in $T^{-1}(U)$; then $T(x) \cdot U$ is non-empty and from (a) there is a neighborhood V of x so that for each x' in V , $T(x') \cdot U$ is non-empty; that is, $V \subset T^{-1}(U)$ and hence $T^{-1}(U)$ is open.

(b) \rightarrow (c): Let $C \subset T(\bar{X})$ be closed; then $T(\bar{X}) - C$ is open and since $T^{-1}(T(\bar{X}) - C) = \bar{X} - T^+(C)$, $\bar{X} - T^+(C)$ is open and $T^+(C)$ is closed.

(c) \rightarrow (a): Let x be in \bar{X} , V open in $T(\bar{X})$, and $V \cap T(x)$ non-empty; then $T(\bar{X}) - V$ is closed in $T(\bar{X})$, $T^+(T(\bar{X}) - V)$ is closed in \bar{X} and $U = \bar{X} - T^+(T(\bar{X}) - V)$ is open in \bar{X} . For x' in U , $T(x')$ is not contained entirely in $T(\bar{X}) - V$ and so $T(x') \cap V$ is non-empty. Therefore, U is the required neighborhood of x in the definition of a lsc function.

A real-valued single-valued function $f: \bar{X} \rightarrow R$ is real upper semi-continuous (rusc) [real lower semicontinuous (rlsc)] at a point b in \bar{X} if for each $k > f(b)$ [each $h < f(b)$] there is a neighborhood V of b so that $k > f(x)$ [$h < f(x)$] for each x in V . The next two theorems relate the concepts of usc, lsc, rusc, and rlsc functions in the real-valued case.

If $T: \bar{X} \rightarrow R$ is a real-valued function with bounded point images, define the single-valued functions $U: \bar{X} \rightarrow R$ and $L: \bar{X} \rightarrow R$ as follows:

$$U(x) = \sup T(x) \text{ and } L(x) = \inf T(x).$$

(2.5) Theorem: Let $T: \bar{X} \rightarrow R$ have bounded point images. Then

- (a) if T is usc, U is rusc and L is rlsc;
- (b) if T is lsc, U is rlsc and L is rusc.

Proof: (a) Let x be in \bar{X} , $h > U(x)$, $k < L(x)$ and a and b be such that $h > b > U(x)$ and $k < a < L(x)$; then (a, b) is a neighborhood of $T(x)$ and since T is usc there is a V in $N^*(x)$ so that $T(V) \subset (a, b)$. Then for each x' in V , $U(x') \leq b < h$ and $L(x') \geq a > k$ which means that U is rusc and L is rlsc at x .

(b) Let x be in \bar{X} , $k > L(x)$, $h < U(x)$ and a and b such that $k > a > L(x)$, $h < b < U(x)$. Then (b, ∞) and $(-\infty, a)$ are neighborhoods of $U(x)$ and $L(x)$. Since T is lsc there is a V in $N^*(x)$ so that x' in V

implies that $T(x') \cap (b, \infty)$ is non-empty and $T(x') \cap (-\infty, a)$ is non-empty; that is, for each x' in V , $U(x') > h$ and $L(x') < k$. Therefore U is rlsc and L is rusc at x .

(2.6) Corollary: If T is continuous, U and L are continuous.

(2.7) Theorem: Let $T: \bar{X} \rightarrow R$ have continuum-valued point images. Then

(a) if U is rusc and L is rlsc, T is usc;

(b) if L is rusc and U is rlsc, T is lsc.

Proof: (a) Let x be in \bar{X} , $T(x) = [a, b]$, and N be in $N^*([a, b])$; then N contains an interval (c, d) where $c < a \leq b < d$. Since U is rusc there is an M_1 in $N^*(x)$ so that x' in M_1 implies that $U(x') < d$; since L is rlsc there is an M_2 in $N^*(x)$ so that x'' in M_2 implies that $L(x'') > c$. Then $M = M_1 \cdot M_2$ is in $N^*(x)$ so that for x' in M , $c < L(x') \leq U(x') < d$; that is, $T(x) \subset (c, d) \subset N$ and so T is usc at x .

(b) Let x be in \bar{X} , $T(x) = [a, b]$, and V be an open set so that $T(x) \cdot V$ is not empty. If a is not equal to b choose a point p in $T(x) \cdot V$ so that p is not a and p is not b and choose h and k so that $a < h < p < k < b$. Since U is rlsc there is an M_1 in $N^*(x)$ so that for x' in M_1 , $U(x') > k$. Since L is rusc there is an M_2 in $N^*(x)$ so that for x'' in M_2 , $L(x'') < h$. Then $M = M_1 \cdot M_2$ is in $N^*(x)$ so that for y in M , $U(y) > k > h > L(y)$; hence p is in $T(y) \cdot V$, which is not empty, and T is lsc at x .

If $a = b$ there is an interval $I = (h, k) \subset V$ so that $T(x)$ is in I . Choose M as above. Here $T(x)$ is in $T(y) \cdot V$ for each y in M and so T is lsc at x .

(2.8) Corollary: The function T is continuous if U and L are continuous.

An interesting extension theorem is a consequence of the last two theorems. It makes use of, and is a generalization of, the Tietze Extension Theorem.

(2.9) Theorem: Let \bar{X} be normal, R the real number line and A a closed subset of \bar{X} . For any continuous, continuum-valued function $T:A \rightarrow R$ there is a function $S:\bar{X} \rightarrow R$ which is continuous and continuum-valued and so that $S|A = T$.

Proof: Since T is continuous and continuum-valued, the functions U and L are continuous, by Theorem (2.5). The Tietze Extension Theorem assures the existence of functions $U':\bar{X} \rightarrow R$ and $L':\bar{X} \rightarrow R$ so that U' and L' are continuous and $U'|A = U$, $L'|A = L$.

Let $m(x) = \max \{U'(x), L'(x)\}$ and $l(x) = \min \{U'(x), L'(x)\}$ for all x in \bar{X} . Then m and l are continuous and $m(x) \geq l(x)$ for all x in \bar{X} . Define $S(x) = [l(x), m(x)]$; then S is continuous from Theorem (2.7) and $S|A = T$.

3. Decompositions I

A decomposition D^* of a set \bar{X} is a collection of non-empty subsets of \bar{X} whose union is \bar{X} and so that the pairwise intersection of distinct members of D^* is empty. A decomposition of a topological space is usc (lsc) if for any closed (open) subset A of \bar{X} the union of all of the members of D^* intersecting A is closed (open).

Every ssv surjection $T:\bar{X} \rightarrow \bar{Y}$ defines decompositions of \bar{X} and \bar{Y} in the following way:

$$D^*(\bar{X}) = \{T^{-1}(y): y \text{ in } \bar{Y}\}, D^*(\bar{Y}) = \{T(x): x \text{ in } \bar{X}\}.$$

For instance, to see that $D^*(\bar{Y})$ is a decomposition recall the definition of a ssv function: if $T(x) \cdot T(x')$ is non-empty then $T(x) = T(x')$. Since T is ssv if and only if T^{-1} is ssv, $D^*(\bar{X})$ is a decomposition for the same reasons as $D^*(\bar{Y})$.

The next theorem relates the ideas of upper and lower semicontinuity for functions and decompositions. A ssv function is quasi-compact if the image of every closed (open) inverse set is closed (open) in the range. A surjection is doubly quasi-compact if both it and its inverse are quasi-compact.

(3.1) Theorem: Let $T: \bar{X} \rightarrow \bar{Y}$ be a ssv doubly quasi-compact surjection. Then T is usc (lsc) if and only if $D^*(\bar{Y})$ is usc (lsc).

Proof: Suppose T is usc and let C be a closed subset of \bar{Y} ; then $T^{-1}(C)$ is closed since T is usc and $TT^{-1}(C)$ is closed since T is quasi-compact. But $TT^{-1}(C) = \Sigma\{T(x): x \text{ in } T^{-1}(C)\} = \Sigma\{T(x): T(x) \cdot C \text{ is not empty}\}$, which is the union of the set of all elements of $D^*(\bar{Y})$ intersecting C . Therefore $D^*(\bar{Y})$ is usc.

Conversely, suppose $D^*(\bar{Y})$ is usc and let C be a closed subset of \bar{Y} . Then $TT^{-1}(C)$ is closed since $D^*(\bar{Y})$ is usc and $T^{-1}TT^{-1} = T^{-1}$ since T is ssv, so that $T^{-1}TT^{-1}(C) = T^{-1}(C)$. Since T^{-1} is quasi-compact, $T^{-1}(TT^{-1}(C))$ is closed and thus $T^{-1}(C)$ is closed. Hence T is usc.

To prove the assertion for the lsc case, replace usc with lsc and closed with open in the above.

(3.2) Remark: Whyburn has proven similar results in his paper on retracting multifunctions [9], to be discussed later on.

4. k-Spaces

A topological space \bar{X} is a k-space when a subset of \bar{X} is closed if its intersection with every closed compact set K is closed in K . If \bar{X} is Hausdorff and either locally compact or first countable, then \bar{X} is a k-space [4, p. 502].

Let $T: \bar{X} \rightarrow \bar{Y}$ be a function from a topological space \bar{X} to a topological space \bar{Y} . The following are two properties which T may or may not satisfy:

$P_1(T)$: $T^{-1}(y)$ is closed in \bar{X} for each y in \bar{Y} ;

$P_2(T)$: for any filterbase $\{F_i\}$ of compact sets of \bar{X} , $T(\Pi F_i) = \Pi T(F_i)$.

Lin and Soniat [6] have characterized Hausdorff k-spaces by showing that \bar{X} is a Hausdorff k-space if and only if $P_1(T)$ and $P_2(T)$ are equivalent for single-valued functions. The next two theorems are extensions of their results to include all functions. The proofs are very much the same as their proofs.

(4.1) Theorem: If \bar{X} is a Hausdorff space, $P_1(T)$ implies $P_2(T)$.

Proof: It follows from Theorem (1.3) (b) that $T(\Pi B^*) \subset \Pi T(B^*)$, where B^* is a filterbase of compact sets and $\Pi B^* = \Pi \{B: B \text{ is in } B^*\}$.

Now let y be in $\Pi T(B^*)$; then $y \in T(B)$ and $T^{-1}(y) \cdot B$ is not empty for each B in B^* . Let $A^* = \{B \cdot T^{-1}(y): B \text{ is in } B^*\}$; A^* is a filterbase by Lemma (1.8) and each A in A^* is compact since each B in B^* is compact and $T^{-1}(y)$ is closed. Thus A^* has a cluster point x in A for each A in A^* since each A is closed. Therefore x is in $T^{-1}(y) \cdot B$ for every B in B^* and so x is in B for every B in B^* . Then x is in ΠB^* and $T(x) \subset T(\Pi B^*)$. This, and the reverse inclusion above,

show that $P_2(T)$ holds.

(4.2) Theorem: Let \bar{X} be Hausdorff. \bar{X} is a k-space if and only if $P_1(T)$ is equivalent to $P_2(T)$ for all T on \bar{X} to Hausdorff \bar{Y} .

Proof: Cohen [2] has shown that \bar{X} is a Hausdorff k-space if and only if it is the quotient space of a locally compact Hausdorff space, say Z . Let $p: Z \rightarrow \bar{X}$ denote the natural projection. Suppose $P_1(T)$ is false, i.e., there exists an element y in \bar{Y} so that $T^{-1}(y)$ is not closed in \bar{X} . Then $p^{-1}(T^{-1}(y))$ is not closed in Z and so there is an element z in the closure of $p^{-1}T^{-1}(y)$ so that y is not in $T(p(z))$. Since Z is a locally compact Hausdorff space there is a filterbase E^* of compact neighborhoods of z so that $\Pi E^* = \{z\}$. Let $F^* = p(E^*)$; then F^* is a filterbase of compact subsets of \bar{X} so that $\Pi F^* = \Pi p(E^*) = p(\Pi E^*)$ from Theorem (4.1) for p . Thus $\Pi F^* = \{p(z)\}$. Hence $T(\Pi F^*) = T(p(z))$; but $\Pi T(F^*)$ contains y and so $T(\Pi F^*)$ does not equal $\Pi T(F^*)$; i.e., $P_2(T)$ is false. By Theorem (4.1), $P_1(T)$ and $P_2(T)$ are equivalent.

If \bar{X} is not a k-space, Lin and Soniat have exhibited a function T (single-valued) for which $P_1(T)$ is not true and $P_2(T)$ is true. This completes the proof of the theorem.

CHAPTER III

DECOMPOSITIONS AND QUASI-COMPACTNESS

In the past the idea of a decomposition generated on a topological space has apparently only been studied for ssv functions. Whyburn [9] has analyzed the ssv surjection $T:\bar{X} \rightarrow \bar{Y}$ in terms of the decomposition spaces $D^*(\bar{X})$ and $D^*(\bar{Y})$ and the natural mappings (continuous single-valued function) $P:\bar{X} \rightarrow D^*(\bar{X})$ and $Q:\bar{Y} \rightarrow D^*(\bar{Y})$ defined by $P(x) = T^{-1}T(x)$ and $Q(y) = TT^{-1}(y)$. The functions P and Q are quasi-compact and if T and T^{-1} are quasi-compact the spaces $D^*(\bar{X})$ and $D^*(\bar{Y})$ are topologically equivalent.

The objectives of this chapter are to extend Whyburn's analysis to include all functions, define quasi-compactness for all functions so as to be consistent with the definition for ssv functions and Whyburn's analysis, and extend some results which have apparently only been known for ssv or single-valued functions.

5. Decompositions II

(5.1) Definition: Let $T:\bar{X} \rightarrow \bar{Y}$ be a function. Define $DT:P(\bar{X}) \rightarrow P(\bar{Y})$ for subsets A of \bar{X} by $DT(A) = T(A) \circ T(\bar{X} - A)$, and $D^{*'} to be that collection of non-empty subsets of \bar{X} for which $DT(A)$ is empty. Define D^* to be the collection of all minimal members of $D^{*'}$ with the partial ordering defined by set inclusion.$

The elements of $D^{*'}$ are the non-empty subsets A of \bar{X} having the property that if $T^{-1}(y) \circ A$ is non-empty then $T^{-1}(y)$ is contained in A .

(5.2) Theorem: Let I be an indexing set and A, A_i in $D^{*'} for all i in I . Then for any function $T: \bar{X} \rightarrow \bar{Y}$,$

- (a) $\bar{X} - A$ is in $D^{*'} if A is not \bar{X} ,$
- (b) $T^{-1}T(A) = A,$
- (c) $T(\bar{X} - A) = T(\bar{X}) - T(A),$
- (d) $\prod A_i$ is in $D^{*'} if $\prod A_i$ is not empty,$
- (e) $\sum A_i$ is in $D^{*'},$ and
- (f) $A - A_i$ is in $D^{*'} if $A - A_i$ is not empty.$

Proof: (a) Since A is in $D^{*'}, DT(\bar{X} - A) = T(A) \cdot T(\bar{X} - A) = DT(A)$, which is empty, so $\bar{X} - A$ is in D^{*}' .

(b) Let x be in $T^{-1}T(A)$; then $T(x) \cdot T(A)$ is not empty and, since $T(A) \cdot T(\bar{X} - A)$ is empty, x is not in $\bar{X} - A$; i.e., x is in A . Therefore $T^{-1}T(A) \subseteq A$; but the reverse inclusion is always true so that $T^{-1}T(A) = A$.

(c) Since A is in D^{*}' , $T(\bar{X} - A) = \{y \text{ in } \bar{Y}: T^{-1}(y) \cdot (\bar{X} - A) \text{ is not empty}\} = T(\bar{X}) - \{y \text{ in } \bar{Y}: T^{-1}(y) \subseteq A\} = T(\bar{X}) - T(A)$.

(d) Suppose $\prod A_i = B$ is not in D^{*}' ; then for some x in $\bar{X} - B$, $T(x) \cdot T(B)$ is not empty. Also, x is in $\bar{X} - A_j$ for some j and so $T(x) \cdot T(B) \subseteq T(\bar{X} - A_j) \cdot T(A_j) = DT(A_j)$ since $T(B) \subseteq T(A_j)$ for any j . This contradicts the fact that A_j is in D^{*}' .

(e) Let $C = \sum A_i$ and y be in $T(C)$; then y is in $T(A_k)$ for some k and hence y is not in $T(\bar{X} - A_k)$ since $DT(A_k)$ is empty. Therefore y is not in $T(\bar{X} - C)$, since $T(\bar{X} - C) \subseteq T(\bar{X} - A_k)$, and $DT(C)$ is empty.

(f) Since both A and A_i are in D^{*}' , so is $A \cdot (\bar{X} - A_i) = A - A_i$ from part (d) and part (a).

(5.3) Theorem: D^* is a decomposition of \bar{X} .

Proof: For A_1, A_2 in D^* , if $B = A_1 \cdot A_2$ is non-empty B is in $D^{*'}$ and contained in A_1 and A_2 . Since A_1 and A_2 are minimal, $A_1 = B = A_2$. To show that $\Sigma D^* = \bar{X}$, let x be in \bar{X} and A' be the intersection of all members of $D^{*'}$ which contain x . From Theorem (5.2) (d), A' is in $D^{*'}$; let A be any other member of $D^{*'}$. If x is in A , $A' \subset A$; if x is not in A , x is in $\bar{X} - A$ and $A' \subset \bar{X} - A$ so that $A \cdot A'$ is empty. Therefore A' is minimal, and in D^* .

(5.4) Remark: If T is ssv the elements of D^* are just point inverses; these are the elements of the natural decomposition for such functions so that D^* is a generalization of $D^*(\bar{X})$.

(5.5) Definition: D^* will be called the natural decomposition of \bar{X} induced by T . Let D^* have the quotient topology and let P be the projection of \bar{X} onto D^* .

It is well known [9, p. 345] that the decomposition D^* is upper (lower) semicontinuous if and only if the mapping P is closed (open). For the ssv case an equivalent condition for the decomposition to be usc is that T be reflexive closed; that is $T^{-1}T(C)$ is closed for all closed sets C in \bar{X} [3, p. 690]. This is not true for arbitrary functions, as is shown in the following example.

(5.6) Example: Let $\{x_i\}$ be any sequence of distinct real numbers converging to p and let $\bar{X} = \bar{Y} = p + \Sigma x_i$. Define $T: \bar{X} \rightarrow \bar{Y}$ by $T(x_i) = \{x_i, x_{i+1}\}$ for all $i = 1, 2, \dots$ and $T(p) = p$. Let C in \bar{X} be closed; C is finite, or infinite containing p . $T^{-1}T(p)$ is then finite, or infinite

containing p so that \mathbb{T} is reflexive closed. However, $D^* = \{\Sigma x_i, p\}$ and, for any i , $P^{-1}P(x_i) = \bar{X} - p$ which is not closed so that D^* is not usc.

It will be shown that there is an analogous condition which reduces to that of being reflexive closed in the ssv case. Some preliminary ideas must be considered before stating the condition.

The function T^{-1} defines sets corresponding to $D^{*'} and D^* , say $E^{*'}$ and E^* . It turns out, as one would expect by considering the ssv case, that $E^{*' = T(D^{*'})$ and $E^* = T(D^*)$.$

(5.7) Theorem: If $T: \bar{X} \rightarrow \bar{Y}$ is a surjection, $E^{*' = T(D^{*'})$ and $E^* = T(D^*)$.

Proof: Since ΣD^* covers \bar{X} and T is a surjection, $\Sigma T(D^*)$ covers \bar{Y} . Now $DT^{-1}(T(A)) = T^{-1}T(A) \cdot T^{-1}(\bar{Y} - T(A)) = A \cdot (\bar{X} - T^{-1}T(A)) = A \cdot (\bar{X} - A)$ which is empty for A in $D^{*'}$, where the necessary algebra comes from Theorem (5.2); thus $T(A)$ is in $E^{*'}$. Now let $DT^{-1}(B)$ be empty and $A = T^{-1}(B)$; then $T(A) \cdot T(\bar{X} - A) = TT^{-1}(B) \cdot T(\bar{X} - T^{-1}(B)) = B \cdot (\bar{Y} - B)$ which is empty so A is in $D^{*'}$. Also $T(A) = TT^{-1}(B) = B$ so that $T(D^{*' = E^{*'}$.

Now let $B = T(A)$ for A in D^* ; it will be shown that B is minimal. Let B' be in $E^{*'}$; then $B' = T(A')$ for some A' in $D^{*'}$. If $B \cdot B'$ is non-empty, $T(A) \cdot T(A')$ is non-empty. Let y be in $T(A) \cdot T(A')$; then $T^{-1}(y) \subset A$ and $T^{-1}(y) \subset A'$ so that $A \cdot A'$ is non-empty. Therefore, since A is in D^* , $A \subset A'$ and $B = T(A) \subset T(A') = B'$.

To see that any element B in E^* is the image of something in D^* , let $A = T^{-1}(B)$; then $T(A) = TT^{-1}(B) = B$ from Theorem (5.2) (b) as applied to T^{-1} . Since B is in $E^{*'}$, A is in $D^{*'}$ from the previous part. Let A' be in $D^{*'}$ and $A \cdot A'$ be non-empty; then $T(A) \cdot T(A')$ is non-empty and by the minimality of $T(A)$, $T(A) \subset T(A')$, and $A = T^{-1}T(A) \subset T^{-1}T(A') = A'$.

Therefore A is in D^* and the theorem is proved.

(5.8) Definition: Let $T: \bar{X} \rightarrow \bar{Y}$ be a function. For any x in \bar{X} the order of x , $O(x)$, is the smallest positive integer n so that $(TT^{-1})^{n-1}T(x)$ is in E^* . If there is no such integer then $O(x) = \infty$.

The order of T , $O(T)$, is the smallest positive integer n so that $O(x) \leq n$ for all x in \bar{X} . If there is no such integer then $O(T) = \infty$.

(5.9) Theorem: Let $T: \bar{X} \rightarrow \bar{Y}$ be a surjection; then for $B = (TT^{-1})^{n-1}T(p)$, $B = TT^{-1}(B)$ if and only if $O(p) \leq n$.

Proof: If $O(p) \leq n$, B is in E^* and $TT^{-1}(B) = B$ from Theorem (5.2) (b). To prove the converse it will first be shown that if $B = TT^{-1}(B)$ then B is a member of $E^{*'} and then that it is a member of E^* .$

It is necessary first to show that for any two subsets of \bar{Y} , A and C , $T^{-1}(A) \cdot T^{-1}(C)$ is empty if and only if $A \cdot TT^{-1}(C)$ is empty. To see this, $T^{-1}(A) \cdot T^{-1}(C)$ is empty if and only if for each x in A , $T^{-1}(x) \cdot T^{-1}(C)$ is empty, which is true if and only if for each y in A , y is not in $TT^{-1}(C)$; i.e., $A \cdot TT^{-1}(C)$ is empty.

Now $B \cdot (\bar{Y} - B) = TT^{-1}(B) \cdot (\bar{Y} - B)$ is empty which implies that $T^{-1}(B) \cdot T^{-1}(\bar{Y} - B)$ is empty from above with $B = C$ and $A = \bar{Y} - B$. Thus $DT^{-1}(B)$ is empty and B is in $E^{*'}$.

Since p is in A for some A in D^* , $B = (TT^{-1})^{n-1}T(p) \subset (TT^{-1})^{n-1}T(A) = T(A) = G$ which is in E^* from Theorem (5.7). By the minimality of G , $B = G$ and so B is in E^* .

An immediate consequence of the definition is

(5.10) Theorem: The function $T: \bar{X} \rightarrow \bar{Y}$ is ssv if and only if $O(T) = 1$.

Proof: If T is ssv, $T(x)$ is in E^* for all x in \bar{X} and hence $O(T) = 1$. Conversely, if $O(T) = 1$, $T(x)$ is in E^* for all x in \bar{X} and, since E^* is a decomposition, $T(x) \cdot T(y)$ non-empty implies that $T(x) = T(y)$; i.e., T is ssv.

It is now possible to give the condition which will ensure that the decomposition D^* is usc.

(5.11) Theorem: Let $T: \bar{X} \rightarrow \bar{Y}$ be a function and $O(T) = n$. Then the decomposition D^* is usc if and only if S^n is closed, where $S = T^{-1}T$.

Proof: For any x in \bar{X} , $(TT^{-1})^{n-1}T(x)$ is in E^* since $O(T) = n$ and thus, by Theorem (5.7), $S^n(x) = T^{-1}(TT^{-1})^{n-1}T(x)$ is in D^* . Also, for any A in D^* and subset C of \bar{X} so that $A \cdot C$ is non-empty, for x in $A \cdot C$, $A = S^n(x) \subset S^n(A \cdot C) \subset S^n(A) = A$; that is $A = S^n(A \cdot C) = S^n(A)$. Therefore $S^n(C) = S^n[\Sigma\{A \cdot C: A \text{ in } D^* \text{ and } A \cdot C \text{ non-empty}\}]$

$$\begin{aligned}
 &= \Sigma\{S^n(A \cdot C): A \text{ in } D^* \text{ and } A \cdot C \text{ non-empty}\} \\
 &= \Sigma\{S^n(A): A \text{ in } D^* \text{ and } A \cdot C \text{ non-empty}\} \\
 &= \Sigma\{A: A \text{ in } D^* \text{ and } A \cdot C \text{ non-empty}\} \\
 &= P^{-1}P(C), \text{ where } P \text{ is the projection mapping.}
 \end{aligned}$$

Thus, if D^* is usc, $P^{-1}P(C)$ is closed for C closed and hence S^n is closed. If S^n is closed then $P^{-1}P(C)$ is closed for C closed and D^* is usc.

(5.12) Corollary: Let $T: \bar{X} \rightarrow \bar{Y}$ be a function and $O(T) = n$. Then if T is reflexive closed the decomposition D^* is usc.

Proof: If T is reflexive closed, $(T^{-1}T)$ is closed and hence S^n is closed.

Note that in Example (5.6) $O(T) = \infty$ since for any x_1 and any n ,

$(TT^{-1})^n T(x_i)$ is not equal to $\bar{X} - p$.

6. Quasi-compact Functions

Quasi-compact functions play an important role in the theory of decompositions. For ssv functions, requiring that the image of a closed inverse set be closed is equivalent to requiring that the image of an open inverse set be open; this is not true, in general, for functions which are not ssv, as is shown by the next example.

(6.1) Example: Let $\bar{X} = \bar{Y} = [0,1]$ with the usual topology; let $T: \bar{X} \rightarrow \bar{Y}$ be defined by $T(x) = \bar{Y} - \{0.50\}$ if x is not 0.50 and $T(0.50) = (0.25, 0.75)$. For $C = \{0.50\}$ in \bar{Y} , $TT^{-1}(C) = (0.25, 0.75)$ which is not closed while $T^{-1}(C) = \{0.50\}$ is closed. The only open inverse sets are \bar{X} and $\bar{X} - \{0.50\}$ and $T(\bar{X}) = \bar{Y}$, $T(\bar{X} - \{0.50\}) = \bar{Y} - \{0.50\}$, both open sets.

Therefore, for arbitrary functions, the image of open inverse sets being open does not force the image of closed inverse sets to be closed. This problem can be alleviated by restricting those inverse sets to be considered in the definition of quasi-compactness. This is done in the following definition in such a way as to reduce to the previous definition in the case of ssv functions.

(6.2) Definition: The function $T: \bar{X} \rightarrow \bar{Y}$ is quasi-compact if and only if $T(A)$ is closed in $T(\bar{X})$ for each closed A in $D^{*'}.$

(6.3) Theorem: The function $T: \bar{X} \rightarrow \bar{Y}$ is quasi-compact if and only if $T(A)$ is open in $T(\bar{X})$ for each open A in $D^{*'}.$

Proof: Suppose T is quasi-compact and A (not \bar{X}) in $D^{*'}$ is open; then

$\bar{X} - A$ is in $D^{*'} by Theorem (5.2) (a) and $\bar{X} - A$ is closed. Thus $T(\bar{X} - A) = T(\bar{X}) - T(A)$ from Theorem (5.2) (c). Since $T(\bar{X} - A)$ is closed in $T(\bar{X})$, $T(A)$ is open in $T(\bar{X})$.$

To prove the converse, replace open with closed and closed with open in the above.

(6.4) Definition: Let $T: \bar{X} \rightarrow \bar{Y}$ be a function, x in \bar{X} , and $A(x)$ the element in D^* containing x . The function $t: \bar{X} \rightarrow \bar{Y}$ is defined by $t(x) = T(A(x))$.

Note that $T(\bar{X}) = t(\bar{X})$ and for ssv functions $T(x) = t(x)$.

(6.5) Theorem: For any subset D' of \bar{X} , D' is a member of $D^{*'}$ if and only if it is the union of the members of some subset of D^* . In particular, if D' is in $D^{*'}$, $D' = \Sigma\{D \text{ in } D^*: D \cdot D' \text{ is non-empty}\}$.

Proof: Suppose D' is in $D^{*'}$; then for every x in D' , x is in $D(x)$ for some D in D^* since D^* is a decomposition, and $D(x)$ is contained in D' since $D(x)$ is minimal. Therefore D' is contained in $\Sigma\{D: D \cdot D' \text{ is non-empty}\}$. The reverse inclusion is always true by the minimality condition.

If D' is the union of the members of some subset of D^* , D' is in $D^{*'}$ by Theorem (5.2) (e).

(6.6) Theorem: Let $T: \bar{X} \rightarrow \bar{Y}$ be a function. Then

(a) t is ssv, and

(b) T is quasi-compact if and only if t is quasi-compact.

Proof: (a) Let x, x' be in \bar{X} ; then $t(x)$ and $t(x')$ are in E^* . Since E^* is a decomposition either $t(x) = t(x')$ or $t(x) \cdot t(x')$ is empty; that is, t is ssv.

(b) Since t is ssv, the natural decomposition of \bar{X} imposed by t is $\{t^{-1}t(x):x \in \bar{X}\} = d^*$. It will be shown that d^* coincides with D^* and hence, from Theorem (6.5), $d^{*'} will equal $D^{*'}$. It will then be shown that $t(A)$ equals $T(A)$ for any A in $D^{*'}$ and thus for any closed A in $D^{*'}$, proving part (b).$

Note that, for p, q in \bar{X} , if $T(A(p)) = T(A(q))$ then $A(p) = T^{-1}T(A(p)) = T^{-1}T(A(q)) = A(q)$. Then $t^{-1}t(p) = \{q \text{ in } \bar{X}: t(p) = t(q) \text{ is non-empty}\} = \{q \text{ in } \bar{X}: t(p) = t(q)\} = \{q \text{ in } \bar{X}: T(A(p)) = T(A(q))\} = \{q \text{ in } \bar{X}: A(p) = A(q)\} = A(p)$, so that $d^* = D^*$.

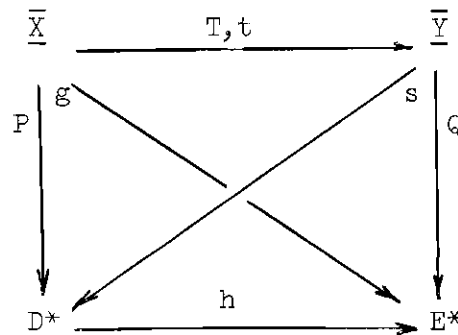
Since $t(A) = T(A)$ for every element A of D^* , and every member of $D^{*'}$ is the union of a subset of the elements of D^* , $t(A') = T(A')$ for any A' in $D^{*'}$. This proves (b).

For any two topological spaces, \bar{X} and \bar{Y} , let $R^*(\bar{X}, \bar{Y})$ be the collection of all functions on \bar{X} to \bar{Y} , and $S^*(\bar{X}, \bar{Y})$, which is a subset of $R^*(\bar{X}, \bar{Y})$ be the set of all ssv functions on \bar{X} to \bar{Y} ; then $R: R^* \rightarrow S^*$ defined by $R(T) = t$ is a surjection. The single-valued function R thus defines an equivalence relation on the set of all functions given by $T_1 \sim T_2$ if and only if $R(T_1) = R(T_2)$. From the last theorem, either all members of any equivalence class are quasi-compact or they are all not quasi-compact, depending upon their representative in S^* . The function R is itself quasi-compact since it is a retraction [9, Theorem (4.5), p. 346].

Now let $P: \bar{X} \rightarrow D^*$ and $Q: \bar{Y} \rightarrow E^*$ be the natural mappings for the decompositions of \bar{X} and \bar{Y} defined by the surjection T . Define $h: D^* \rightarrow E^*$ by $h(D) = QtP^{-1}(D)$, $g: \bar{X} \rightarrow E^*$ by $g(x) = hP(x)$, and $s: \bar{Y} \rightarrow D^*$ by $s(y) = h^{-1}Q(y)$; then P, Q, h, g , and s are single valued surjections, as pointed out by Whyburn [9]. The following theorem is a direct analogue of Theorem (6.1)

in [9]; here the requirement that T be ssv is not needed. The proof is exactly the same since T is quasi-compact if and only if t is quasi-compact.

The following commutative diagram is useful for understanding the functions h , g , and s and the theorem following:



(6.7) Theorem: Let $T: \bar{X} \rightarrow \bar{Y}$ be a surjection; then

(a) h is one-to-one and quasi-compact when T is quasi-compact; also h^{-1} is quasi-compact when T^{-1} is quasi-compact;

(b) h is a homeomorphism when both T and T^{-1} are quasi-compact. Conversely, if h is a homeomorphism both T and T^{-1} are quasi-compact;

(c) the function g is quasi-compact when T is quasi-compact and continuous when T^{-1} is quasi-compact;

(d) the function s is quasi-compact when T^{-1} is quasi-compact and continuous when T is quasi-compact.

The following theorem of Whyburn [9, Theorem 6.2, p. 348] analyzes the ssv representative t of the function T .

(6.8) Theorem: For any ssv doubly quasi-compact surjection $t: \bar{X} \rightarrow \bar{Y}$ there exists a topological space Z and a pair of continuous single-valued

quasi-compact surjections $P: \bar{X} \rightarrow Z$, $Q: \bar{Y} \rightarrow Z$ satisfying $t = Q^{-1}P$, $t^{-1} = P^{-1}Q$. Thus t is closed (open) if and only if P is closed (open); equivalently, t is usc (lsc) if and only if Q is closed (open).

Conversely, any pair of quasi-compact single-valued continuous surjections $P: \bar{X} \rightarrow Z$, $Q: \bar{Y} \rightarrow Z$ defines a ssv doubly quasi-compact function under the definition $t(x) = Q^{-1}P(x)$, x in \bar{X} , which in turn generates a pair P' and Q' equivalent to P and Q .

Theorems (6.9), (6.10), (6.11), and (6.13) are generalizations of Theorems (4.1), (4.2), (4.4), and (4.5) in [9]. In every case, Whyburn's results require that the function be ssv.

(6.9) Theorem: Every open or closed function is quasi-compact.

Proof: If the image of every open (or closed) set is open (closed), the same holds true for those in D^{*} .

(6.10) Theorem: Every usc compact-valued function $T: \bar{X} \rightarrow \bar{Y}$ is quasi-compact, for \bar{X} compact and \bar{Y} Hausdorff.

Proof: Let C be a closed subset of \bar{X} ; then C is compact and $T(C)$ is compact since compactness is invariant under usc compact-valued functions [8, Corollary A₂, p. 1497]. Since \bar{Y} is Hausdorff, $T(C)$ is closed and hence T is closed and so quasi-compact.

For any surjection $T: \bar{X} \rightarrow \bar{Y}$, a set S in \bar{X} is a cross-section for T if and only if $T(S) = \bar{Y}$.

(6.11) Theorem: Let the surjection $T: \bar{X} \rightarrow \bar{Y}$ be quasi-compact on some cross-section S for T ; then T is quasi-compact.

Proof: Let A be in $D^{*'}; then $A \cdot S = S \cdot T^{-1}(B) = (T|S)^{-1}(B)$ for some B in $E^{*'}$. If A is a closed set, $A \cdot S$ is closed in S . Also $A \cdot S$ is in $(D|S)^{*'}, where $(D|S)^{*'} = \{C \text{ in } S : D(T|S)(C) \text{ is empty}\}$. To see this, $(T|S)(A \cdot S) \cdot (T|S)(S - A \cdot S) \subset T(A) \cdot T(\bar{X} - A) = DT(A)$, which is empty. Thus, since $(T|S)$ is quasi compact, $T(A \cdot S)$ is closed in \bar{Y} . However, $T(A \cdot S) = T(A)$; for let y be in $T(A)$. Then $T^{-1}(y) \cdot A$ is empty and so $T^{-1}(y) \subset A$ since A is in $D^{*'}; but $S \cdot T^{-1}(y)$ is non-empty since S is a cross-section and so there is an x in $S \cdot T^{-1}(y) \subset S \cdot A$, which means that $T^{-1}(y) \cdot (S \cdot A)$ is not empty and y is in $T(S \cdot A)$. The reverse inclusion is always true; i.e., $T(S \cdot A) \subset T(A)$. Therefore $T(A)$ is closed in \bar{Y} and T is quasi-compact.$$$

A surjection $T: \bar{X} \rightarrow \bar{Y} \subset \bar{X}$ is retracting if y is in $T(y)$ for each y in \bar{Y} .

(6.12) Theorem: Let $T: \bar{X} \rightarrow \bar{Y} = \bar{X}$ be a retracting surjection; then

- (a) T^{-1} is retracting,
- (b) $D^* = E^*$ and $D^{*'} = E^{*'}$,
- (c) $T(A) = T^{-1}(A) = A = TT(A)$ for each A in $D^{*'}$.

Proof: (a) Let y be in \bar{Y} ; then y is in \bar{X} and y is in $T(y)$. For any z and x , z is in $T(x)$ if and only if x is in $T^{-1}(z)$ so y is in $T(y)$ if and only if y is in $T^{-1}(y)$; i.e., T^{-1} is retracting.

(b) Let A be in D^* and $B = T(A)$ be in E^* . If x is in A , x is in $T(x) \subset T(A) = B$; also, for y in B , y is in $T^{-1}(y) \subset T^{-1}(B) = T^{-1}T(A) = A$ from Theorem (5.2)(b), so $A = B$ and $D^* = E^*$. If A' is in $D^{*'}$ then A' is a union of members of $D^* = E^*$, from Theorem (6.5), and so A' is in $E^{*'}$; a similar argument for B' in $E^{*'}$ shows that $D^{*' = E^{*'}$.

(c) Let A be in D^* ; then, from the proof of part (b), $T(A) = B = A$; a similar argument yields $T^{-1}(A) = A$, and since $T(A) = A$, $TT(A) = T(A)$.

(6.13) Theorem: Every retracting function is quasi-compact.

Proof: Let $T: \bar{X} \rightarrow \bar{Y}$ be retracting and let $S = T|_Y$; then $S: \bar{Y} \rightarrow \bar{Y}$ is a retracting surjection. Moreover, $S(A) = A$ from Theorem (6.12) so that S is quasi-compact. However, \bar{Y} is a cross-section for T since $T(\bar{Y}) = \bar{Y}$, so T is quasi-compact by Theorem (6.11).

Theorem (6.14) and Theorem (6.18) are generalizations of Theorems 1, 2, 3, and 5 of E. Duda [3]; Duda's theorems are stated as corollaries to (6.14) and (6.18). Recall that a mapping is a continuous single-valued function; since there are no continuity conditions in (6.14) and (6.18) Duda's results hold for any single-valued functions satisfying the remaining hypotheses.

(6.14) Theorem: Let \bar{X} be a Hausdorff space and $T: \bar{X} \rightarrow \bar{Y}$ a function so that $T^{-1}T(x)$ is compact for each x in \bar{X} . Then (a) implies (b); if \bar{X} is a k -space, (b) implies (a); if $O(T)$ is finite (a) implies (c); if \bar{X} is a k -space and $O(T)$ is finite, (b) implies (c).

- (a) T is reflexive closed;
- (b) T is reflexive compact;
- (c) the natural decomposition of \bar{X} is usc.

Proof: (a) \rightarrow (b): Define $S: \bar{X} \rightarrow \bar{X}$ by $S(x) = T^{-1}T(x)$. Then x is in $T^{-1}T(y)$ if and only if $T(x) \cdot T(y)$ is non-empty, which is true if and only if y is in $T^{-1}T(x)$; i.e., x is in $S(y)$ if and only if y is in $S(x)$. Therefore $S^{-1}(x) = \{y \in \bar{X}: x \text{ is in } S(y)\} = \{y \in \bar{X}: y \text{ is in } S(x)\} = S(x)$.

Thus any property which holds under S also holds under S^{-1} . Since T is reflexive closed S is closed and hence usc; S has compact point images by hypothesis and therefore [8, Corollary A₂, p. 1497] S preserves compactness and is thus compact.

(b) \rightarrow (a): The function S above now preserves compactness and thus so does S^{-1} ; $S(x)$ is closed for all x in \bar{X} and so from the theorem in [8, Corollary C'₆, p. 1499] which states that if \bar{X} and \bar{Y} are Hausdorff spaces and \bar{Y} is a k -space, any compact function with closed point values is closed, S is closed and thus T is reflexive closed.

(a) \rightarrow (c): This is a consequence of Corollary (5.11).

(b) \rightarrow (c): This follows from (b) \rightarrow (a) and (a) \rightarrow (c).

(6.15) Corollary: Let \bar{X} be a Hausdorff k -space and f a mapping of \bar{X} onto \bar{Y} . If f is reflexive compact then f generates an upper semi-continuous decomposition.

(6.16) Corollary: Let f be a mapping with compact point inverses of a Hausdorff space \bar{X} onto a space \bar{Y} . If f generates an usc decomposition then f is reflexive compact.

(6.17) Corollary: Let f be a mapping with compact point inverses of a Hausdorff k -space \bar{X} into a space \bar{Y} . The mapping f generates an usc decomposition if and only if f is reflexive compact.

Note that if T in Theorem (6.14) is ssv and \bar{X} is a k -space, (a), (b), and (c) are equivalent.

A function $T: \bar{X} \rightarrow \bar{Y}$ is semi-closed if the image of every compact subset of \bar{X} is closed in \bar{Y} . Thus any continuous single-valued function

onto a Hausdorff space is semi-closed with a semi-closed inverse.

(6.18) Theorem: If \bar{X} is a Hausdorff k-space and $T: \bar{X} \rightarrow \bar{Y}$ is a quasi-compact, reflexive compact surjection with finite order, and T^{-1} is semi-closed, then T is compact.

Proof: Since T is reflexive compact, the decomposition D^* of \bar{X} is usc by Theorem (6.14). From Theorem (6.7) (d) the single-valued surjection $S: \bar{Y} \rightarrow D^*$ defined by $s(y) = h^{-1} q(y)$ is continuous, since T is quasi-compact, and thus preserves compactness.

For any point A in D^* , $P^{-1}(A) = (T^{-1}T)^n(x)$ for some n and any x in A and since T is reflexive compact P has compact point inverses. Since D^* is usc, P is closed and hence compact [8, Corollary A₃, p. 1497]. Thus $t^{-1}(K)$ is compact for any compact subset of \bar{Y} since $t^{-1}(K) = P^{-1}s(K)$.

Since T^{-1} is semi-closed, $T^{-1}(K)$ is closed; also $T^{-1}(K) \subset t^{-1}(K)$ since $T(x) \cdot K$ is non-empty which implies that $t(x) \cdot K$ is non-empty because $T(x) \subset t(x)$. Therefore $T^{-1}(K)$ is compact and T is a compact function.

(6.19) Corollary: Let $f: \bar{X} \rightarrow \bar{Y}$ be an onto mapping, where \bar{X} is a Hausdorff k-space. If f is quasi-compact and reflexive compact, then f is compact.

Proof: The order of f is one, and hence finite, and every continuous function on a Hausdorff range is such that f^{-1} is semi-closed.

CHAPTER IV

SEMICONINUITY AND COMPACTNESS

Every single-valued continuous function is usc, lsc, preserves compactness and, with a Hausdorff range, is semi-closed with a semi-closed inverse. In this chapter these concepts and some others which arise naturally will be studied, especially in relation to continuity and compactness.

G. T. Whyburn [8] has done some work in this direction for functions and R. V. Fuller [4] for single-valued functions. Whyburn uses what he calls directed families of sets; these are filterbases. Fuller uses nets in his work and defines a concept called subcontinuity; the generalization and extension of his results is carried out using filterbases, and a concept called semicontinuity is defined which serves essentially the same purposes as subcontinuity.

7. B-closed and Semi-closed Functions

(7.1) Definition: The graph of a function $T:\bar{X} \rightarrow \bar{Y}$ is the subset

$$\text{gr}(T) = \{(x,y): y \in T(x), x \in \bar{X}\} \text{ of } \bar{X} \times \bar{Y}.$$

(7.2) Definition: (Berge [1]) A function $T:\bar{X} \rightarrow \bar{Y}$ is B-closed if and only if for each x in \bar{X} and y in $\bar{Y} - T(x)$ there are neighborhoods U of x and V of y so that $T(U) \cdot V$ is empty.

(7.3) Definition: The function $T:\bar{X} \rightarrow \bar{Y}$ is semicontinuous at the point x in \bar{X} if and only if for every filterbase $B^* \rightarrow x$ every filterbase finer

than $T(B^*)$ has a cluster point in \underline{Y} .

(7.4) Definition: A subset A of \bar{X} is conditionally compact if every filter on A has a cluster point.

(7.5) Remark: If \bar{X} is a Hausdorff space, a set A is conditionally compact if and only if the closure of A is compact. Also, A is a compact subset of \bar{X} if and only if every filter on A has a cluster point in A . It should also be noted that in (7.4) and the comments preceeding in (7.5), the word filterbase can be substituted for the word filter.

(7.6) Theorem: The following conditions on the function $T: \bar{X} \rightarrow \bar{Y}$ are equivalent:

- (a) T has a closed graph.
- (b) T is B-closed.
- (c) For each x in \bar{X} and filterbase $B^* \rightarrow x$, every cluster point of $T(B^*)$ is in $T(x)$.

Proof: (a) \rightarrow (b): Let y be in $\bar{Y} - T(x)$; i.e., (x, y) is not in $\text{gr}(T)$. Let $N = U \times V$ be an open base neighborhood of (x, y) missing $\text{gr}(T)$; then the projections $p_x(N) = U$ and $p_y(N) = V$ are open neighborhoods of x and y and $T(U) \cdot V$ is empty; that is, T is B-closed.

(b) \rightarrow (c): Let y be a cluster point of $T(B^*)$, where B^* is a filterbase converging to x ; then every open neighborhood of y intersects each member of $T(B^*)$ nonvacuously. Since $B^* \rightarrow x$, every open neighborhood of x contains some member of B^* , and hence the image of any open neighborhood of x intersects every open neighborhood of y . Since T is B-closed, this can be true only if y is in $T(x)$.

(c) \rightarrow (a): Let (x, y) be in $\overline{X} \times \overline{Y} - \text{gr}(T)$; then y is in $\overline{Y} - T(x)$.

Let N^* be an open base for the neighborhood filter of x (that is, $N^*(x)$ is the set of all subsets of \overline{X} which contain an element of N^*); then $N^* \rightarrow x$ and so any cluster point of $T(N^*)$ is in $T(x)$. Therefore there is some neighborhood V of y which misses some element of $T(N^*)$; say $T(U)$; i.e., for each z in U , $T(z) \cdot V$ is empty. Let $N = U \times V$; then N is a neighborhood of (x, y) which misses $\text{gr}(T)$ and so $\text{gr}(T)$ is closed.

(7.7) Theorem: Let $T: \overline{X} \rightarrow \overline{Y}$ be a function. If $T^{-1}(y)$ is closed for each y in \overline{Y} , (a) \rightarrow (b); if \overline{X} a Hausdorff space, (b) \rightarrow (a).

(a) T is semi-closed.

(b) For every filterbase $F^* \rightarrow x$ in \overline{X} whose elements are compact sets, the set of cluster points of $T(F^*)$ is contained in $T(x)$.

Proof: (a) \rightarrow (b): Let $F^* \rightarrow x$ be a filterbase of compact sets and let y be a cluster point of $T(F^*)$; then y is in $\text{cl}(T(F))$ (the closure of $T(F)$) for each F in F^* . Since each $T(F)$ is closed in \overline{Y} by (a), y is in $T(F)$ and $F \cdot T^{-1}(y)$ is non-empty for each F in F^* . Let N be in $N^*(x)$; then there is an element F' in F^* so that F' is a subset of N , since $F^* \rightarrow x$. Therefore $F' \cdot T^{-1}(y)$ is non-empty and so $N \cdot T^{-1}(y)$ is non-empty; i.e., x is in $\text{cl}(T^{-1}(y))$. Since $T^{-1}(y)$ is closed by hypothesis, x is in $T^{-1}(y)$ which means that y is in $T(x)$.

(b) \rightarrow (a): Suppose T is not semi-closed; then there is a compact subset K of \overline{X} so that $T(K)$ is not closed in \overline{Y} and a filterbase $F^* \rightarrow y$, y not in $T(K)$, where F^* is in $T(K)$. Then $T^{-1}(F^*)$ is a filterbase in \overline{X} by Lemma (1.6) and, since $T^{-1}(F) \cdot K$ is non-empty for every F in F^* , there is a filterbase B^* in K finer than $T^{-1}(F^*)$, by Lemma (1.8).

Since K is compact B^* has a cluster point x in K and there is a filterbase $D^* \rightarrow x$ finer than B^* , where D^* is in K .

Now let D^{**} be the filterbase whose elements are the closures of elements of D^* . Since K is closed, because \bar{X} is Hausdorff, and D^* is in K , D^{**} is in K ; also $D^{**} \rightarrow x$. To see this, let N be in $N^*(x)$. Since K is a compact Hausdorff space, K is regular and there is an M in $N^*(x)$ so that $\text{cl}(M) \cdot K$ is contained in $N \cdot K$. Since $D^* \rightarrow x$ and D^* is in K , there is a D in D^* so that D is a subset of M . Therefore $\text{cl}(D) \subset \text{cl}(M) \cdot K \subset N \cdot K \subset N$.

By Lemma (1.7) there is a filterbase G^* which is finer than $T(D^*)$ and F^* . Since $F^* \rightarrow y$, $G^* \rightarrow y$; also $T(D^*)$ is finer than $T(D^{**})$ so, since $G^* \rightarrow y$ and since y is a cluster point of $T(D^*)$, y is a cluster point of $T(D^{**})$. Now D^{**} is made up of compact sets and y is a cluster point of $T(D^{**})$, but y is not in $T(x)$ since D^{**} is in K and so $T(D^{**})$ is in $T(K)$, while y is not in $T(K)$. That is, condition (b) does not hold. This proves the theorem.

(7.8) Theorem: Let $T: \bar{X} \rightarrow \bar{Y}$ be a function; then (a) \rightarrow (b). If T has closed point inverses and \bar{X} and \bar{Y} are Hausdorff and either locally compact or first countable, (b) \rightarrow (a).

(a) T is B-closed.

(b) T is semi-closed.

Proof: (a) \rightarrow (b): Let K be a compact subset of \bar{X} and y in $\bar{Y} - T(K)$. For each x in K , y is not in $T(x)$ since $T^{-1}(y) \cdot K$ is empty; therefore there are open neighborhoods $U(x)$ of x and $V(y)$ of y so that $T(U(x)) \cdot V(y)$ is empty since T is B-closed. The set $\{U(x): x \text{ in } K\}$ is

an open cover of K and hence there is a finite subcover, say U_1, U_2, \dots, U_n . The set $V = V_1 \cdot V_2 \cdot \dots \cdot V_n$ is open and, since $T(U_i) \cdot V_i$ is empty for all $i = 1, 2, \dots, n$, $T(U_i) \cdot V$ is empty. Equivalently, $T^{-1}(V) \cdot U_i$ is empty for $i = 1, 2, \dots, n$ and so $T^{-1}(V) \cdot K$ is empty; equivalently $V \cdot T(K)$ is empty. Thus $\bar{Y} - T(K)$ is open and $T(K)$ is closed in \bar{Y} .

(b) \rightarrow (a): Let x be in \bar{X} , y in $\bar{Y} - T(x)$ and let U be a neighborhood of x missing $T^{-1}(y)$.

First countable case: Let $\{U_i\}$ and $\{V_i\}$ be countable bases for $N^*(x)$ and $N^*(y)$ so that U_i is a subset of U for all i . If T is not B -closed at x , for each k there is an x_k in U_k and a y_k in $T(x_k) \cdot V_k$. Since $A = x + \Sigma x_k$ is compact, $T(A)$ is closed; but $\{y_k\} \rightarrow y$ so that y is in $T(A)$ since Σy_k is contained in $T(A)$, and thus $T^{-1}(y) \cdot A$ is non-empty. However, A is a subset of U and $U \cdot T^{-1}(y)$ is empty, leading to a contradiction. Thus T is B -closed.

Locally compact case: Let W be a compact neighborhood of x contained in U ; then $T(W)$ is closed and misses y . Let V be a neighborhood of y missing $T(W)$; then W and V satisfy the requirements for T to be B -closed.

Berge [1] calls closed a function which has been defined as B -closed. Here a function is closed if and only if it takes closed sets onto closed sets in its range.

The following example exhibits a B -closed function which is not closed; Kuratowski [5, Remark, p. 175] has an example of a function which is closed but not B -closed. Corollary (7.12) states some conditions under which a closed function is B -closed. Theorem (8.1) implies that if T^{-1} is semicontinuous and B -closed T is closed.

(7.9) Example: Let $\bar{X} = [0,1)$, $\bar{Y} = [0,1]$ and define $T: \bar{X} \rightarrow \bar{Y}$ by $T(x) = x$ if x is not zero and $T(0) = \{0,1\}$. To see that T is B-closed:

(a) for $x = 0$, y not 0 or 1, take a (0.5) y neighborhood of x and y ;

(b) for x not zero, x not y , take a $(0.5)|x - y|$ neighborhood of x and y .

These neighborhoods satisfy the requirements for B-closedness.

Since $T([0.5,1)) = [0.5,1)$, T is not closed.

Let $T: \bar{X} \rightarrow \bar{Y}$ be a function; then \bar{Y} is T-regular if for x in \bar{X} and y in $\bar{Y} - T(x)$ there are disjoint open neighborhoods of y and $T(x)$. Note that \bar{Y} is T-regular if either \bar{Y} is regular and T has closed point values, or \bar{Y} is Hausdorff and T has compact point values. Thus, for any single-valued function $f: \bar{X} \rightarrow \bar{Y}$, if \bar{Y} is Hausdorff it is f -regular.

(7.10) Theorem: Let $T: \bar{X} \rightarrow \bar{Y}$ be usc with \bar{Y} T-regular; then T is B-closed.

Proof: Let x be in \bar{X} , y in $\bar{Y} - T(x)$ and V and N be disjoint open neighborhoods of y and $T(x)$. Since T is usc there is a U in $N^*(x)$ so that $T(U) \subset N$, and so $T(U) \cdot V$ is empty and T is B-closed at x .

(7.11) Corollary: (Whyburn [8, Theorem C, p. 1498]) if $T: \bar{X} \rightarrow \bar{Y}$ is an usc function and \bar{Y} is T-regular (in particular if \bar{Y} is Hausdorff and T is compact valued) then T is semi-closed.

Proof: Since T is B-closed, Theorem (7.8) insures that T is semi-closed.

(7.12) Corollary: If $T: \bar{X} \rightarrow \bar{Y}$ is a closed surjection and \bar{X} is T^{-1} -regular, then T is B-closed.

Proof: If T is closed, T^{-1} is usc and the theorem guarantees that T^{-1} is B-closed. Therefore, T^{-1} has a closed graph, from (7.6); but T is a surjection so that T has a closed graph and hence T is B-closed.

(7.13) Corollary: (Kuratowski [5, Theorem 1, p. 175]). Suppose that \bar{Y} is a regular space and T an usc function. Then T has a closed graph.

8. The Semicontinuities

Any continuous single-valued function on a Hausdorff space is semi-closed with a semi-closed inverse, as was pointed out earlier. Fuller [4] has proved that a necessary and sufficient condition for a single-valued function on a Hausdorff space to be continuous is that it have a closed graph and be subcontinuous. Here the question is: if T is a semi-closed or B-closed function, what additional hypotheses are necessary to insure that T is usc or lsc?

It will be seen that the answer to this question lies in the study of the location of the cluster points of the images of convergent filter-bases in \bar{X} . In this respect it is more enlightening to study functions than single-valued functions because it can be seen exactly which part of continuity, usc or lsc, requires which conditions.

The following theorem was proved by Whyburn [8, Theorem A, p. 1497] in a slightly different form.

(8.1) Theorem: If T is a semicontinuous, B-closed function on \bar{X} to \bar{Y} , then T is usc and compact valued. If \bar{Y} is Hausdorff the converse is true.

Proof: Suppose \bar{Y} is Hausdorff and T is usc and compact valued; then T is B-closed by Theorem (7.10) since \bar{Y} is then T -regular.

It will next be shown that T is semicontinuous. Let x be in \overline{X} , $B^* \rightarrow x$ a filterbase and N^* a filterbase finer than $T(B^*)$. Since T is B -closed the only possible cluster points of $T(B^*)$ lie in $T(x)$, by Theorem (7.6), and, since N^* is finer than $T(B^*)$, all the cluster points of N^* are also cluster points of $T(B^*)$. Thus the only possible cluster points of N^* are in $T(x)$.

Suppose there are none; then for each y in $T(x)$ there is an open $U(y)$ in $N^*(y)$ and $N(y)$ in N^* so that $U(y) \cdot N(y)$ is empty. Since $T(x)$ is compact and $\{U(y): y \text{ in } T(x)\}$ is an open cover, there is a finite subcover of $T(x)$, $U(y_1), U(y_2), \dots, U(y_n)$. Then $N' = \cap N(y_i)$ contains an element N in N^* and for $U = \sum U(y_i)$, $N \cdot U$ is empty. Since T is usc, there is an open set V so that x is in V and $T(V) \subset U$; thus $T(V) \cdot N$ is empty. By Lemma (1.7) there is a filterbase C^* finer than $T^{-1}(N^*)$ and B^* . Since C^* is finer than B^* , $C^* \rightarrow x$; since C^* is finer than $T^{-1}(N^*)$ there is some C in C^* so that $C \subset T^{-1}(N)$. Since $T(V) \cdot N$ is empty, $V \cdot T^{-1}(N)$ is empty and so $V \cdot C$ is empty; thus C^* could not converge to x . This contradiction proves that T is semicontinuous.

Now suppose that T is B -closed and semicontinuous; it will first be shown that T is compact valued. Let x be in \overline{X} and A^* be a filter in $T(x)$. Then A^* has a base B^* finer than the filterbase $\{T(x)\}$. Since the filterbase $\{x\} \rightarrow x$ and T is semicontinuous, B^* has a cluster point which is in $T(x)$ since $T(x)$ is closed, which follows from the B -closedness of T . Thus so does A^* and $T(x)$ is compact.

To see that T is usc, let B be a closed subset of \overline{Y} and $A = T^{-1}(B)$. Suppose p is in $\text{cl}(A) - A$ and $M^*(p) = \{N \cdot A: N \text{ is in } N^*(p)\}$ where $N^*(p)$ is an open base for the neighborhood filter of p . Then

$M^*(p) \rightarrow p$. Since T is semicontinuous the set of cluster points of $T(M^*(p))$ is non-empty and since T is B -closed the set of cluster points of $T(M^*(p))$ lies in $T(p)$.

Now let $P^* = \{T(M) \cdot B : M \text{ is in } M^*(p)\}$; P^* is filterbase by Lemma (1.8) and P^* is finer than $T(M^*(p))$. Hence P^* has a cluster point in $T(p)$. But $T(p)$ is in the complement of B in \bar{Y} and $(\bar{Y} - B) \cdot P$ is empty for each P in P^* so, since B is closed, each r in $T(p)$ has a neighborhood missing B and hence each element of P^* . Thus P^* could not have a cluster point in $T(p)$, a contradiction. Thus $\text{cl}(A) - A$ is empty and A is closed; that is, T is usc.

The hypothesis of compact point images cannot be excluded in the previous theorem as is shown by the next example. Here T is an usc function which is not semicontinuous.

(8.2) Example: Let $\bar{X} = [0, 1]$, $\bar{Y} = [0, 1)$ and $T: \bar{X} \rightarrow \bar{Y}$ be defined by $T(x) = [x, 1)$ if x is not 1 and $T(1) = [0, 1)$.

The function T is usc at $x = 1$ since \bar{Y} is the only neighborhood of $T(1)$ and so contains the image of any neighborhood of $x = 1$. However, if $N^* = \{(x, 1) : 0 < x < 1\}$, $N^* \rightarrow 1$ but for each y in \bar{Y} , every element of $T(N^*)$ corresponding to N in N^* for which $x > y$ misses some neighborhood of y and so no y in \bar{Y} is a cluster point of $T(N^*)$ and T is not semicontinuous at $x = 1$.

The following theorem was proved by Smithson [7]:

(8.3) Theorem: (Smithson) A surjection $T: \bar{X} \rightarrow \bar{Y}$ is lsc if and only if for each x in \bar{X} and filterbase $A^* \rightarrow x$, each y of $T(x)$ is a cluster point of $T(A^*)$.

This leads to:

(8.4) Theorem: The surjection $T:\bar{X} \rightarrow \bar{Y}$ is B-closed and lsc if and only if for each x in \bar{X} and filterbase $A^* \rightarrow x$ the set of cluster points of $T(A^*)$ is exactly $T(x)$.

Proof: First suppose that T is B-closed and lsc; then, from Theorem (7.6), all the cluster points of $T(A^*)$ are contained in $T(x)$. Since T is lsc, from Theorem (8.3), each point in $T(x)$ is a cluster point of $T(A^*)$. Thus the set of cluster points of $T(A^*)$ is exactly $T(x)$.

The converse also follows from (7.6) and (8.3).

(8.5) Theorem: Let $f:\bar{X} \rightarrow \bar{Y}$ be a single-valued surjection and \bar{Y} a Hausdorff space; then the following conditions are equivalent:

- (a) f is continuous.
- (b) For any filterbase $A^* \rightarrow x$ in \bar{X} , the set of cluster points of $f(A^*)$ is exactly $f(x)$.
- (c) For any filterbase $A^* \rightarrow x$ in \bar{X} , $f(x)$ is a cluster point of $f(A^*)$.
- (d) For any filterbase $A^* \rightarrow x$ in \bar{X} , $f(A^*)$ converges.

Prcof: (a) \rightarrow (b): Since f is usc and compact valued, it is B-closed by Theorem (8.1). From Theorem (8.4), since f is lsc, the set of cluster points of $f(A^*)$ is exactly $f(x)$.

(b) \rightarrow (c): This is obvious.

(c) \rightarrow (a): Here (c) implies that f is lsc, from Theorem (8.3), and hence continuous.

(a) \rightarrow (d): Since $A^* \rightarrow x$, A^* is finer than the neighborhood filter

of x , considering the neighborhood filter as a filterbase. Since the image of the neighborhood filter converges, so does $f(A^*)$.

(d) \rightarrow (a): Since $N^*(x)$ is a filterbase converging to x , $f(N^*(x))$ converges, say to p . Therefore every neighborhood of p contains a member of $f(N^*(x))$ and hence $f(x)$, since x is in N for each N in $N^*(x)$; but \bar{Y} is Hausdorff so unless $p = f(x)$ there would be a neighborhood of p missing $f(x)$, proving that $p = f(x)$. Thus f is continuous.

Fuller [4] has an analogue to the (d) part of (8.5); it states that f is continuous if the image of every convergent net converges.

The following two examples show that semicontinuity does not imply upper or lower semicontinuity, and that continuity does not imply semicontinuity.

(8.6) Example: The function T_1 is not usc and T_2 is not lsc but they are both semicontinuous.

Let \bar{X} be any topological space containing a subset, S , which is neither open nor closed, and \bar{Y} any compact topological space containing an open proper subset U and a closed proper subset C . Define $T_1: \bar{X} \rightarrow \bar{Y}$ and $T_2: \bar{X} \rightarrow \bar{Y}$ as follows:

$$T_1(x) = C, x \text{ in } S \text{ and } T_1(x) = \bar{Y} - C, x \text{ in } \bar{X} - S;$$

$$T_2(x) = U, x \text{ in } S \text{ and } T_2(x) = \bar{Y} - U, x \text{ in } \bar{X} - S.$$

Then $T_1^{-1}(C) = S$ and $T_2^{-1}(U) = S$ so that T_1 is not usc and T_2 is not lsc. Since \bar{Y} is compact, T_1 and T_2 are necessarily semicontinuous, from the definition.

(8.7) Example: Let \bar{X} be any topological space and \bar{Y} any topological space which is not compact. Define $T: \bar{X} \rightarrow \bar{Y}$ by $T(x) = \bar{Y}$ for all x in \bar{X} . For any

subset, S , of \bar{Y} , $T^{-1}(S) = \bar{X}$ which is both open and closed, so that T is continuous. For any filterbase A^* in \bar{X} , $T(A^*) = \{\bar{Y}\}$; since \bar{Y} is not compact there is a filterbase B^* in \bar{Y} with no cluster point and B^* is finer than $\{\bar{Y}\}$, so T is not semicontinuous.

9. Semicontinuity and Compactness

The concept of semicontinuity, as was mentioned earlier, plays an important role in the discussion of compactness preserving functions. It is easy to see that any function into a compact space is semicontinuous. The next few theorems investigate how semicontinuous functions operate on compact sets.

(9.1) Theorem: The function $T: \bar{X} \rightarrow \bar{Y}$ is semicontinuous if and only if the inverse image of any filterbase in $T(\bar{X})$ with no cluster point in \bar{Y} has no cluster point.

Proof: Suppose T is semicontinuous and A^* is a filterbase in $T(\bar{X})$ with no cluster point in \bar{Y} . If $T^{-1}(A^*)$ has a cluster point, say p , there is a filterbase B^* finer than $T^{-1}(A^*)$ which converges to p ; by Lemma (1.7) there is a filterbase C^* which is finer than both A^* and $T(B^*)$. Since C^* is finer than $T(B^*)$ and $B^* \rightarrow x$, C^* has a cluster point in \bar{Y} , by semicontinuity. Thus, since C^* is finer than A^* , A^* has a cluster point in \bar{Y} , which contradicts the original assumption. Therefore, $T^{-1}(A^*)$ has no cluster point.

Now suppose that T is not semicontinuous; then there is a filterbase $A^* \rightarrow p$ in \bar{X} and a filterbase B^* in $T(\bar{X})$ which is finer than $T(A^*)$ but has no cluster point in \bar{Y} . By Lemma (1.7) there is a filterbase C^* which is finer than both $T^{-1}(B^*)$ and A^* . Since C^* is finer than A^* ,

$C^* \rightarrow p$, and since C^* is finer than $T^{-1}(B^*)$, p is a cluster point of $T^{-1}(B^*)$. To see this, let U be in $N^*(p)$; there is an element C in C^* so that C is contained in U . If D is in $T^{-1}(B^*)$ and $D \cdot C$ is empty, since C^* is finer than $T^{-1}(B^*)$ there would be an element C' contained in D so that C' is in C^* ; that is $C \cdot C'$ would be empty. This is impossible since C^* is a filterbase. Therefore B^* is a filterbase in $T(\bar{X})$ with no cluster point in \bar{Y} but $T^{-1}(B^*)$ has a cluster point, the negation of the condition in the statement of the theorem.

(9.2) Corollary: If the function $T: \bar{X} \rightarrow \bar{Y}$ is semicontinuous the image of every compact set is conditionally compact.

Proof: Let K be a compact subset of \bar{X} and A^* a filterbase in $T(K)$; then $T^{-1}(A^*) \cdot K = \{T^{-1}(A) \cdot K : A \text{ in } A^*\}$ is a filterbase in K , by Lemma (1.8), and so $T^{-1}(A^*) \cdot K$ has a cluster point in K . Therefore $T^{-1}(A^*)$ has a cluster point since $T^{-1}(A^*) \cdot K$ is finer than $T^{-1}(A^*)$ and so A^* must have a cluster point in \bar{Y} , by the theorem; that is, $T(K)$ is conditionally compact.

(9.3) Corollary: A surjection T on a compact space \bar{X} to a space \bar{Y} is semicontinuous if and only if \bar{Y} is compact.

Proof: If \bar{Y} is compact, T is semicontinuous. Conversely, since \bar{X} is compact $T(\bar{X}) = \bar{Y}$ is conditionally compact, from (9.2). Since every filterbase in $T(\bar{X})$ has a cluster point in $\bar{Y} = T(\bar{X})$, \bar{Y} is compact.

(9.4) Theorem: Let $T: \bar{X} \rightarrow \bar{Y}$ be a function which takes every compact subset of \bar{X} onto a conditionally compact subset of \bar{Y} . If \bar{X} is locally compact (every x in \bar{X} has a compact neighborhood), T is semicontinuous.

Proof: Let F^* be a filterbase in \bar{X} converging to x and B^* a filterbase finer than $T(F^*)$. There is a filterbase A^* in \bar{X} which is finer than both F^* and $T^{-1}(B^*)$, by Lemma (1.7). Let K be a compact neighborhood of x ; then $C^* = \{A \cdot K : A \text{ in } A^*\}$ is a filterbase finer than A^* by Lemma (1.8). Again, by Lemma (1.7) there is a filterbase D^* in \bar{Y} which is finer than both $T(C^*)$ and B^* . Since C^* is a filterbase in K , $T(C^*)$ is in $T(K)$ and, since D^* is finer than $T(C^*)$, every element of D^* intersects $T(K)$. Hence there is a filterbase E^* finer than D^* so that E^* is in $T(K)$, by Lemma (1.8).

Now $T(K)$ is conditionally compact by hypothesis, so E^* has a cluster point in \bar{Y} . Since E^* is finer than D^* and D^* is finer than B^* , E^* is finer than B^* and B^* has a cluster point in \bar{Y} ; i.e., T is semicontinuous.

(9.5) Corollary: For a locally compact domain, a function is semicontinuous if and only if the image of every compact set is conditionally compact.

The following example shows that semicontinuity is not equivalent to taking compact sets onto conditionally compact sets. It also illustrates that a function which preserves compactness of sets is not necessarily semicontinuous (see Theorem (9.14)).

(9.6) Example: This is an example of a function which preserves compactness of sets, and hence takes every compact set onto a conditionally compact set, but which is not semicontinuous.

Let $\bar{X} = \bar{Y} =$ any uncountable set, with the topology on \bar{X} being that consisting of all complements of countable sets, and the discrete topology on \bar{Y} . It will next be shown that the compact subsets of \bar{X} are exactly

the finite subsets of \bar{X} .

Let A be any countably infinite subset of \bar{X} ; then for each a in A the set $S(a) = a + (\bar{X} - A)$ is open and contains a , so that the collection $\{S(a): a \text{ in } A\}$ is an open cover of A which has no finite subcover; i.e., A is not compact. If B is any uncountable subset of \bar{X} , B contains a countable subset, which is thus closed. If B were compact so would the countable subset be compact, which is impossible, so that B is not compact; i.e., the compact subsets of \bar{X} are exactly the finite subsets of \bar{X} .

Now let $I: \bar{X} \rightarrow \bar{Y}$ be the identity function. I preserves compactness since finite sets are compact in \bar{Y} .

To show that I is not semicontinuous, let p be in \bar{X} and $N^*(p) \rightarrow p$. Let $B^* = \{N - \{p\}: N \text{ is in } N^*(p)\}$ and further let B_1, B_2 be in B^* . Then B_1 and B_2 are both complements of countable sets so that $B_1 \cdot B_2$ is also the complement of a countable set. Also, $p + B_1 \cdot B_2$ is a neighborhood of p so that $B_1 \cdot B_2$ is in B^* and B^* is a filterbase which is finer than $N^*(p)$.

Now let y be in \bar{Y} . If $y = p$, y is not a cluster point of B^* in \bar{Y} since the neighborhood $\{y\}$ of y does not intersect any member of B^* in \bar{Y} ; if y is not equal to p the set $\bar{Y} - \{y\}$ is a neighborhood of p and so $\bar{Y} - \{y, p\}$ is in B^* and the neighborhood $\{y\}$ of y does not intersect $\bar{Y} - \{y, p\}$. Thus y is not a cluster point of B^* in \bar{Y} and I is not semicontinuous anywhere on \bar{X} .

(9.7) Example: This is an example of a topological space which is locally compact but not a k -space (recall that any locally compact Hausdorff space is a k -space). It is included to show that even if Theorem (9.4) could be proven with the hypothesis that the domain is

a k -space, (9.4) would not be included.

Let $\underline{\bar{X}} = [0, \infty)$ and the topology T^* defined by adding the point $x = 0$ to each open set of the usual topology and adding the empty set to this collection. The set $\underline{\bar{X}}$ with the usual topology is locally compact; to see that $(\underline{\bar{X}}, T^*)$ is locally compact, use the same compact neighborhoods with the addition of the point $x = 0$.

But $(\underline{\bar{X}}, T^*)$ is not a k -space. The set $\{0\}$ is not closed since $\text{cl}(\{0\}) = \underline{\bar{X}}$, but $\{0\} \cdot K$, where K is any closed compact set, is surely closed since it is empty. This is so since any closed set containing $x = 0$ is not compact since it must be the entire space.

(9.8) Theorem: Let $T: \underline{\bar{X}} \rightarrow \underline{\bar{Y}}$ be a function; if T is semi-closed and if the image of every compact set is conditionally compact, T preserves compactness. If $\underline{\bar{Y}}$ is Hausdorff the converse is also true.

Proof: Let K be a compact subset of $\underline{\bar{X}}$ and F^* a filter in $T(K)$; then F^* has a cluster point, say q , in $\underline{\bar{Y}}$ since $T(K)$ is assumed to be conditionally compact. But $T(K)$ is closed since T is semi-closed, and so q is in $T(K)$ and $T(K)$ is compact.

Conversely, for K a compact subset of $\underline{\bar{X}}$, $T(K)$ is closed since it is a compact subset of a Hausdorff space, and conditionally compact since it is compact.

(9.9) Corollary: If T is semi-closed and semicontinuous, T preserves compactness.

Proof: If T is semicontinuous it takes compact sets onto conditionally compact sets by Corollary (9.2). Now apply (9.8).

(9.10) Corollary: If T is B -closed and semicontinuous, T preserves compactness.

Proof: If T is B -closed it is semi-closed, by Theorem (7.8). Now apply (9.9).

(9.11) Corollary: If T is an usc compact valued function on \bar{X} to \bar{Y} , \bar{Y} Hausdorff, then T preserves compactness.

Proof: Theorem (8.1) insures that T is B -closed and semicontinuous. Corollary (9.10) completes the proof.

(9.12) Corollary: If T is a closed function with compact point inverses on \bar{X} to \bar{Y} , \bar{X} Hausdorff, then T is compact.

Proof: This is Corollary (9.11) applied to T^{-1} .

(9.13) Remark: In every theorem involving functions there is another theorem involving its inverse, as in Corollary (9.12) above.

The following result is a generalization of a result of Fuller [4]; he proved it for a single-valued function $f: \bar{X} \rightarrow \bar{Y}$ using subcontinuity; i.e., for every net $\{x_i\} \rightarrow p$ there is a subset $\{x_{Nb}\}$ of $\{x_i\}$ and a point q in \bar{Y} so that $f(x_{Nb}) \rightarrow q$.

(9.14) Theorem: The function $T: \bar{X} \rightarrow \bar{Y}$ preserves compactness if and only if $T|K: K \rightarrow T(K)$ is semicontinuous for every compact subset K of \bar{X} .

Proof: Let K be a compact subset of \bar{X} . Then $T|K: K \rightarrow T(K)$ is a surjection and thus $T(K)$ is compact if and only if $T|K$ is semicontinuous, from Corollary (9.3). This proves the theorem.

(9.15) Corollary: (Whyburn [8, Corollary C₆, p. 1498]). Let $T: \bar{X} \rightarrow \bar{Y}$ preserve compactness and have closed point inverses. If \bar{X} and \bar{Y} are Hausdorff and \bar{X} is a k-space, T is usc.

Proof: Let C be a closed subset of \bar{Y} and K a compact subset of \bar{X} . Since \bar{X} is a k-space it is sufficient to show that $A = T^{-1}(C) \cdot K$ is closed.

Let $S = T|_K$; then $S: K \rightarrow S(K)$ is semicontinuous by the theorem and semi-closed from Theorem (9.8). Since K is a compact Hausdorff space it is thus consequently locally compact; by Theorem (7.8) S is B-closed and thus, by Theorem (8.1), usc. Now $B = C \cdot T(K)$ is compact and thus closed so that $S^{-1}(B)$ is closed in K and so closed in \bar{X} ; but $S^{-1}(B) = A$, so that T is usc.

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